Final Exam - Review - Problems

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1 Diagonalization

Problem 1

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2 Orthogonality

Problem 2

Apply the Gram-Schmidt process to find an **orthonormal** basis for $W = Span\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, where:

$$\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Problem 3

Find the orthogonal projection of $f(x) = \cos(x)$ on W, where:

$$W = Span\{\sin(x), \sin(2x), \cos(2x)\}$$

with respect to the following inner product:

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)dx$$

3 Symmetric matrices

Problem 4

Find an **orthogonal** matrix P and a diagonal matrix D such that $A = PDP^T$, where:

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

Problem 5

Write the quadratic form $x_1^2 - 6x_1x_2 + 9x_2^2$ without cross-product terms.

4 Vector Spaces

Problem 6

Let $\mathcal{B} = \{e^x, e^x \cos(x), e^x \sin(x)\}$, and let $V = Span(\mathcal{B})$.

Define $T : V \longrightarrow V$ by:

$$T(y) = y' + 2y$$

Find the matrix of T relative to $\mathcal{B}$

Problem 7

Let V be the vector space of 2×2 symmetric matrices. Find a basis for V and the dimension of V.

Problem 8

For the following matrix A, find $Rank(A)$ and a basis for $Row(A), Col(A), Nul(A)$:

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5 True/False Extravaganza!

Problem 9

(a) If $Nul(A) = \{0\}$, then $Rank(A)$ is the number of columns of A.

(b) If A is a 6×8 matrix, then the smallest possible dimension of $Nul(A)$ is 6.

(c) If $dim(V) = 3$ and $T : V \to V$ is one-to-one, then it is also onto.

(d) If Q is an $n \times n$ orthogonal matrix, then $det(Q) = \pm 1$.

(e) If A is symmetric, then eigenvectors corresponding to different eigenvalues are orthogonal.

(f) If W is a subspace of V and $y \in V$, then there is a unique vector $\widetilde{w}$ in W such that $\|y - \widetilde{w}\| \leq \|y - w\|$ for all $w \in W$

(g) If A diagonalizable, then so is A^2

(h) If the characteristic polynomial of A is $(\lambda - 1)^3$, then A has 3 linearly independent eigenvectors.

(i) If A is an orthogonal $n \times n$ matrix, then $Row(A) = Col(A)$

(j) Linear algebra is so much more awesome than differential equations! :)